Problem #1: 

\[ T = Trans(y, 1) \cdot Trans(x, 3) \cdot Rot(z, \frac{\pi}{2}) \]

\[
T = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \cdot \begin{bmatrix}
1 & 0 & 0 & 3 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \cdot \begin{bmatrix}
\cos\left(\frac{\pi}{2}\right) & -\sin\left(\frac{\pi}{2}\right) & 0 & 0 \\
\sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} = \begin{bmatrix}
0 & -1 & 0 & 3 \\
1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
Problem #2:

\[ \mathcal{A}_P = \mathcal{A}_B \cdot \mathcal{B}_P \]

\[ \mathcal{A}_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix} \cdot \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c\theta & -s\theta & 0 \\ c\phi s\theta & c\phi c\theta & -s\theta \\ s\phi s\theta & s\phi c\theta & c\phi \end{bmatrix} \]

Problem #3:

Imagine a Frame \{A\} whose Z axis is aligned with the direction K.

Then the rotation which rotates vectors about K by \( \theta \) could be written as:

\[ R = \mathcal{O}_A \cdot \mathcal{R}(Z_A, \theta) \cdot \mathcal{A}_O \]  \( (1) \)

\( \mathcal{A}_O \) rotates O to coincide with A. Then we rotate around Z axis (which is now coincident with K) and we rotate back to frame O by \( \mathcal{O}_A \).

If we didn’t do that the results would be in A frame not O.

If we write the description of A in O as:

\[ \mathcal{O}_A = \begin{bmatrix} A & D & K_x \\ B & E & K_y \\ C & F & K_z \end{bmatrix} \]

Noting that

\[ \mathcal{A}_O = \left( \mathcal{O}_A \right)^{-1} = \left( \mathcal{O}_A^T \right)^T \]

If we multiply out (1) and then simplify using:

\[ A^2 + B^2 + C^2 = 1, \quad D^2 + E^2 + F^2 = 1, \quad \begin{bmatrix} A \\ B \\ C \end{bmatrix} \cdot \begin{bmatrix} D \\ E \\ F \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} A \\ B \\ C \end{bmatrix} \times \begin{bmatrix} D \\ E \\ F \end{bmatrix} = \begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} \]

we arrive at the expression for \( K \) given in the assignment.

Also see Richard P. Paul, “Robot Manipulators: Mathematics, Programming and Control”, attached at the end.
Problem #4

In method 1 the multiplication of two 3x3 matrices requires 27 multiplications and 18 additions. The computation of $^5R_0$ takes two matrix multiplies or 54 multiplications and 36 additions. The computation of $^6R_0^5P$ is 9 multiplication and 9 additions. Hence in one second this method requires:

\[ 30 \times \text{computation of } ^5R_0 = 30 \times 54 \text{ multiplications} + 30 \times 36 \text{ additions} \]
\[ 100 \times \text{computation of } ^6P = 100 \times 9 \text{ multiplication} + 100 \times 6 \text{ addition} \]

**TOTAL_{method1} = 2520 \text{ multiplications} + 1680 \text{ additions}**

In method 2 computation of $^C R_0^D P$ requires 9 multiplication and 6 addition; likewise the computation of $^B R_0^C P$ and $^A R_0^B P$, for a total of 27 multiplication and 18 addition. These must occur 100 times/sec, so in one second we have:

**TOTAL_{method2} = 27 \times 100 = 2700 \text{ multiplications} + 18 \times 100 = 1800 \text{ additions}**

Therefore method 1 is superior but not by much.

Problem #5.

a) Here are the two Matlab functions:

```matlab
function R=Rot(axis,angle)
% Rot(axis,angle) returns the rotational transformation matrix along the given axis for the given angle. angle is in radians, axis is 'x', 'y' or 'z'.

switch axis
    case {'x'}
        R=[1 0 0;0 cos(angle) -sin(angle);0 sin(angle) cos(angle)];
        return
    case {'y'}
        R=[cos(angle) 0 sin(angle);0 1 0;-sin(angle) 0 cos(angle)];
        return
    case {'z'}
        R=[cos(angle) -sin(angle) 0;sin(angle) cos(angle) 0;0 0 1];
        return
end
error('axis must be x, y or z');

% trans(x,y,z) returns the homogenous transformation matrix describing a translation along the given
```
% displacements

function T = trans(x, y, z)
T = [1 0 0 x;...
     0 1 0 y;...
     0 0 1 z;...
     0 0 0 1];
return

Then the transformation matrices are:

$^0H_1 = \text{trans}(0,1,1) =$

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

$^0H_2 = \text{trans}(-0.5,1.5,1.1) =$

\[
\begin{bmatrix}
1 & 0 & 0 & -0.5 \\
0 & 1 & 0 & 1.5 \\
0 & 0 & 1 & 1.1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

$^0H_3 = \text{trans}(-0.5,1.5,3) \ast \text{rot}(z', \pi/2) \ast \text{rot}(x', \pi) =$

\[
\begin{bmatrix}
0 & 1 & 0 & -0.5 \\
1 & 0 & 0 & 1.5 \\
0 & 0 & -1 & 3 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

$^2H_3 = {^2H_0} {^0H_3} = (^0H_2)^{-1} {^0H_3} =$

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1.9 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
b) Let's find the image coordinates of the corner of the cube at \( P = (0.1,0.1,0.1) \) wrt Frame 2. The coordinates of this point wrt frame 3 can be found from

\[
^3P = ^3H_2 \cdot ^2P = (^2H_3)^{-1} \cdot [0.1 \ 0.1 \ 0.1 \ 1]^T = [0.1 \ 0.1 \ 1.8 \ 1]^T
\]

The mapping from the camera frame (Frame 3) to the sensor plane can be written as:

\[
x_s = f(X)/(Z) \quad y_s = f(Y)/(Z)
\]

Let's say we have a lens system with a focal distance of 30 mm, then:

\[
x_s = 0.03 \times 0.1/1.8 = 0.0016 \text{ [meters]}
\]
\[
y_s = 0.03 \times 0.1/1.8 = 0.0016 \text{ [meters]}
\]

These coordinates are in meters. To change it into pixels you would need to know the size of your sensing elements on the sensor plate (remember that the sensor is an array of small sensing elements). If each element has size \( s_x \) and \( s_y \) in \( x \) and \( y \) directions then the corresponding pixels would be found by:

\[
p_x = x_s/s_x \quad \text{and} \quad p_y = y_s/s_y
\]
*Another Solution for Problem #3*

We will now develop the transformation matrix representing a rotation around an arbitrary vector \( \mathbf{k} \) located at the origin. (See [Hamilton] for a full discussion of this subject.) In order to do this we will imagine that \( \mathbf{k} \) is the z axis unit vector of a coordinate frame \( \mathbf{C} \)

\[
\mathbf{C} = \begin{bmatrix}
    n_x & o_x & a_x & 0 \\
    n_y & o_y & a_y & 0 \\
    n_z & o_z & a_z & 0 \\
    0 & 0 & 0 & 1 \\
\end{bmatrix}
\]  

(1.58)

\( \mathbf{k} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} \)  

(1.59)

Rotating around the vector \( \mathbf{k} \) is then equivalent to rotating around the z axis of the frame \( \mathbf{C} \).

\[
\text{Rot}(\mathbf{k}, \theta) = \text{Rot}\left( \begin{array}{c}
    \mathbf{C} \\
    \text{z, } \theta
\end{array} \right)
\]  

(1.60)

If we are given a frame \( \mathbf{T} \) described with respect to the reference coordinate frame, we can find a frame \( \mathbf{X} \) which describes the same frame with respect to frame \( \mathbf{C} \) as

\[
\mathbf{T} = \mathbf{C} \mathbf{X}
\]  

(1.61)

where \( \mathbf{X} \) describes the position of \( \mathbf{T} \) with respect to frame \( \mathbf{C} \). Solving for \( \mathbf{X} \) we obtain

\[
\mathbf{X} = \mathbf{C}^{-1} \mathbf{T}
\]  

(1.62)

Rotating \( \mathbf{T} \) around \( \mathbf{k} \) is equivalent to rotating \( \mathbf{X} \) around the z axis of frame \( \mathbf{C} \)

\[
\text{Rot}(\mathbf{k}, \theta) \mathbf{T} = \mathbf{C} \text{Rot}(\mathbf{z}, \theta) \mathbf{X}
\]  

(1.63)

\[
\text{Rot}(\mathbf{k}, \theta) \mathbf{T} = \mathbf{C} \text{Rot}(\mathbf{z}, \theta) \mathbf{C}^{-1} \mathbf{T}
\]  

(1.64)
Thus

\[
\text{Rot}(k, \theta) = C \text{Rot}(z, \theta) C^{-1}
\]  

(1.65)

However, we have only \( k \), the \( z \) axis of the frame \( C \). By expanding Equation 1.65 we will discover that: \( C \text{Rot}(z, \theta) C^{-1} \) is a function of \( k \) only.

Multiply \( \text{Rot}(z, \theta) \) on the right by \( C^{-1} \) we obtain

\[
\text{Rot}(z, \theta) C^{-1} = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 & 0 \\
\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
x_n \cos \theta - o_x \sin \theta & n_y \cos \theta - o_y \sin \theta & n_z \cos \theta - o_z \sin \theta & 0 \\
x_n \sin \theta + o_x \cos \theta & n_y \sin \theta + o_y \cos \theta & n_z \sin \theta + o_z \cos \theta & 0 \\
a_x & a_y & a_z & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]  

(1.66)

premultiplying by

\[
C = \begin{bmatrix}
x_n & o_x & a_x & 0 \\
x_y & o_y & a_y & 0 \\
x_z & o_z & a_y & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]  

(1.67)
we obtain $C \text{Rot}(z, \theta)C^{-1} =$

$$
\begin{bmatrix}
  n_x n_x \cos \theta - n_x o_x \sin \theta + n_x o_x \sin \theta + o_x o_x \cos \theta + a_x a_x \\
  n_y n_x \cos \theta - n_y o_x \sin \theta + n_x o_y \sin \theta + o_y o_x \cos \theta + a_y a_x \\
  n_z n_x \cos \theta - n_z o_x \sin \theta + n_x o_z \sin \theta + o_z o_x \cos \theta + a_z a_x \\
  0 \\
  n_x n_y \cos \theta - n_x o_y \sin \theta + n_y o_x \sin \theta + o_y o_x \cos \theta + a_x a_y \\
  n_y n_y \cos \theta - n_y o_y \sin \theta + n_y o_y \sin \theta + o_y o_y \cos \theta + a_y a_y \\
  n_z n_y \cos \theta - n_z o_y \sin \theta + n_y o_z \sin \theta + o_z o_y \cos \theta + a_z a_y \\
  0 \\
  n_x n_z \cos \theta - n_x o_z \sin \theta + n_z o_x \sin \theta + o_z o_x \cos \theta + a_x a_z \\
  n_y n_z \cos \theta - n_y o_z \sin \theta + n_z o_y \sin \theta + o_z o_y \cos \theta + a_y a_z \\
  n_z n_z \cos \theta - n_z o_z \sin \theta + n_z o_z \sin \theta + o_z o_z \cos \theta + a_z a_z \\
  0 \\
\end{bmatrix}
$$

(1.68)

Simplifying, using the following relationships:
the dot product of any row or column of $C$ with any other row or column is zero.
as the vectors are orthogonal:
the dot product of any row or column of $C$ with itself is 1 as the vectors are of unit magnitude:
the $z$ unit vector is the vector cross product of the $x$ and $y$ vectors or

$$
\mathbf{a} = \mathbf{n} \times \mathbf{o}
$$

(1.69)

which has components

$$
\begin{align*}
  a_x &= n_y o_z - n_z o_y \\
  a_y &= n_z o_x - n_x o_z \\
  a_z &= n_x o_y - n_y o_x
\end{align*}
$$

the versine, abbreviated as vers $\theta$, is defined as vers $\theta = (1 - \cos \theta)$. 
\( k_x = a_x, k_y = a_y, \text{ and } k_z = a_z. \)

We obtain \( \text{Rot}(k, \theta) = \)

\[
\begin{bmatrix}
  k_x k_x \text{ vers} \theta + \cos \theta & k_y k_x \text{ vers} \theta - k_z \sin \theta & k_z k_x \text{ vers} \theta + k_y \sin \theta & 0 \\
  k_x k_y \text{ vers} \theta + k_z \sin \theta & k_y k_y \text{ vers} \theta + \cos \theta & k_z k_y \text{ vers} \theta - k_x \sin \theta & 0 \\
  k_x k_z \text{ vers} \theta - k_y \sin \theta & k_y k_z \text{ vers} \theta + k_x \sin \theta & k_z k_z \text{ vers} \theta + \cos \theta & 0 \\
  0 & 0 & 0 & 0
\end{bmatrix} \tag{1.70}
\]

This is an important result and should be thoroughly understood before proceeding further.